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Gauge formulation for higher order gravity

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Abstract. This work is an application of the second order gauge theory for the Lorentz group, where a description of the gravitational interaction is obtained that includes derivatives of the curvature. We analyze the form of the second field strength, $G = \partial F + fAF$, in terms of geometrical variables. All possible independent Lagrangians constructed with quadratic contractions of F and quadratic contractions of G are analyzed. The equations of motion for a particular Lagrangian, which is analogous to Podolsky's term of his generalized electrodynamics, are calculated. The static isotropic solution in the linear approximation was found, exhibiting the regular Newtonian behavior at short distances as well as a meso-large distance modification.

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1 Introduction

Nowadays there are many proposals to modify gravitation in order to solve several problems, as the present day accelerated expansion of the universe [1–6], or to accommodate corrections of a quantum nature that arise from the classical effective backreaction of quantum matter in a curved background [7–10]. The effective action is widely used in quantum field theory as a powerful method of calculation. Podolsky generalized electrodynamics, for instance, can be viewed as an effective description of the quantum correction to the classical Maxwell Lagrangian [11–13].

For gravitation, usually higher orders terms are introduced by means of Lagrangian contributions quadratic in the Riemann tensor and their contractions [14, 15]. This is inspired by one-loop corrections in the Einstein–Hilbert action in the quantized weak field approximation, or in the equivalent Feynman construction of a spin-2 field on a flat Minkowski background [16]. Besides this, at the quantum level, the S matrix for the Einstein theory is finite at one-loop level but diverges at the two-loop order [17], which motivates the introduction of derivative terms in the Riemann tensor for the action [18, 19].

On the other hand, recently was proposed a second order construction of gauge theories based on Utiyama's approach [20], which gives exactly the same correction terms as in Podolsky's electrodynamics, but now arising from the principle of local gauge invariance [21]. Therefore, a connection between quantum corrections and higher order gauge terms in the action was conjectured, which was proved be fulfilled also for the effective Alekseev–Arbuzov–Baikov Lagrangian of the infrared regime of QCD [22].

Here, we analyze the gauge formulation of the gravitational field based on the framework of the second order gauge theory. The simplest gauge group is given by the Lorentz homogeneous group in the context of a Riemannian description of the gravitational field. Since the gauge field is given in such a case by the local spin connection, a higher order in the gauge field naturally involves the derivative of the curvature tensor. In this sense, the actual higher order gravitational Lagrangian should be constructed from invariants using the covariant derivative of the Riemann tensor instead of the usual quadratic terms in the curvature.

The relationship between the algebraic gauge description and the geometrical one is settled by means of the introduction of the tetrad field, and the construction of the covariant derivatives associated with both symmetries: local Lorentz and global diffeomorphic coordinate transformations. We use Latin indexes, $a, b, \ldots$, for the internal Lorentz group and Greek indexes for the tangent space of the space-time manifold.

The paper is structured as follows. In Sect. 2 we review some results relating gauge invariance and gravitation. The field strengths F and G of the second order treat-

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ment are introduced in Sect. 3, where they are also written in their geometrical counterparts: the Riemann curvature tensor and its covariant derivative.

Section 4 deals with the possible quadratic invariants of the type F^2 and G^2. All the possible contractions are studied and only the independent invariants are kept. This counting is made in the same spirit as the systematic selection of the independent Riemann monomials done in [23, 24]. In the following section, Sect. 5, these invariants are shown to satisfy the identity which restricts the theories that may be called of gauge type.

Among all invariants, we select $L_{\mathrm{P}} = \frac{1}{2} h \delta^{\rho} R_{\rho\chi} \delta_{\mu} R^{\mu\chi}$, the Podolsky-like Lagrangian, for calculating the equation of motion of the gravitational field. This higher order gravity application is done in Sect. 6. For this Lagrangian, we calculate the static isotropic solution in the linear regime at Sect. 6.2, finding the regular Newtonian potential at short scales, but with a modified potential at intermediary scales.

Final remarks are given in Sect. 7.

2 Gauge interaction and covariance

In 1956 Utiyama [20] has shown how to implement a gauge description for the gravitational interaction with matter fields $Q^A(x)$ transforming according to

$$\delta Q^A(x) = \frac{1}{2} \varepsilon^{ab}(x) (\Sigma_{ab})^A_B Q^B \,, \tag{1}$$

as an implementation of the local invariance exigency of the action under continuous proper Lorentz transformations, which are characterized by the generators Σ_{ab} satisfying the operation of a typical Lie group,

$$[\Sigma_{ab}, \Sigma_{cd}] = \frac{1}{2} f^{ef}_{ab,cd} \Sigma_{ef} \,, \tag{2}$$

where the

$$f^{ef}_{ab,cd} = \left\{ \left[\eta_{bc} \delta^e_a - \eta_{ac} \delta^e_b \right] \delta^f_d - \left[\eta_{bd} \delta^e_a - \eta_{ad} \delta^e_b \right] \delta^f_c \right\} - e \leftrightarrow f$$

are the structure constants, obeying the Jacobi identity. $\varepsilon^{ab} = -\varepsilon^{ba}$ are the parameters of the local transformation. The capital Latin indexes are for the components of the matter field.

It was clearly shown that one needs to introduce the compensating field $\omega^{ab}_{\mu}(x)$ transforming as a connection,

$$\delta \omega^{ef}_{\mu} = \frac{1}{4} \varepsilon^{ab}(x) f^{ef}_{ab,cd} \omega^{cd}_{\mu} + \partial_{\mu} \varepsilon^{ef}(x) \,. \tag{3}$$

To ensure the covariance under coordinate transformations it was necessary to define a space-time connection whose behavior under infinitesimal diffeomorphisms is

$$\bar{\delta} \Gamma^{\nu}_{\mu\alpha} = \frac{\partial \delta x^{\nu}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\alpha} - \frac{\partial \delta x^{\lambda}}{\partial x^{\mu}} \Gamma^{\nu}_{\lambda\alpha} - \frac{\partial \delta x^{\lambda}}{\partial x^{\alpha}} \Gamma^{\nu}_{\mu\lambda} - \frac{\partial^2 \delta x^{\nu}}{\partial x^{\mu} \partial x^{\alpha}} \,.$$

The invariance of the theory implies that the compensating field must appear through the *gauge covariant derivative*

$$D_{\mu} Q^A \equiv \partial_{\mu} Q^A - \frac{1}{2} \omega^{ab}_{\mu} (\Sigma_{ab})^A_B Q^B \,, \tag{4}$$

i.e.,

$$\delta D_{\mu} Q^A = \frac{1}{2} \varepsilon^{ab} (\Sigma_{ab})^A_B D_{\mu} Q^B \,, \tag{5}$$

and the space-time connection must appear through the space-time covariant derivative,

$$\delta_{\mu} Q^{\lambda\nu} \equiv \partial_{\mu} Q^{\lambda\nu} + \Gamma^{\lambda}_{\mu\beta} Q^{\beta\nu} + \Gamma^{\nu}_{\mu\alpha} Q^{\lambda\alpha} \,, \tag{6}$$

and the total covariant derivative:

$$\nabla_{\mu} Q^{i\nu} = \partial_{\mu} Q^{i\nu} - \omega^{ib}_{\mu} \eta_{bk} Q^{k\nu} + \Gamma^{\nu}_{\mu\alpha} Q^{i\alpha} \,. \tag{7}$$

This total derivative must commute with the mapping to the tangent space of the manifold,[1]

$$Q^{i\mu} \equiv h^{\mu}_j Q^{ij} \,, \quad Q^{ij} = h^j_{\nu} Q^{i\nu} \,, \tag{8}$$

$$\nabla_{\mu} Q^{i\nu} \equiv h^{\nu}_j \nabla_{\mu} Q^{ij} \,, \tag{9}$$

where we have introduced the tetrad field h:

$$h^j_{\nu} h^{\mu}_j = \delta^{\mu}_{\nu} \,, \qquad h^i_{\nu} h^{\nu}_j = \delta^i_j \,,$$
$$g_{\mu\nu} = \eta_{ij} h^i_{\nu} h^j_{\mu} \,, \qquad \eta_{ij} = h^{\mu}_i h^{\nu}_j g_{\mu\nu} \,,$$
$$h = \sqrt{\det h^j_{\mu}} = \sqrt{-g} \,.$$

The definition (9) implies the absolute parallelism of the tetrad:

$$\nabla_{\mu} h^j_{\alpha} \equiv 0 \,, \tag{10}$$

which can be solved for the compensating field,

$$\omega^j_{i\mu} \equiv h^{\alpha}_i \left(\delta_{\mu} h^j_{\alpha} \right)$$

or for the space-time connection,

$$\Gamma^{\nu}_{\mu\alpha} \equiv h^{\nu}_j \left(D_{\mu} h^j_{\alpha} \right) . \tag{11}$$

We will restrict our analysis to a symmetric space-time connection in order to approach the Riemannian description. The extension to the Riemann–Cartan case is quite natural, but it would imply different types of invariants as admissible Lagrangians (see discussion below).

3 Gauge field Lagrangian

The basic hypothesis we will assume is that the Lagrangian for the free gauge potential depends on the

[1] Note that the action of the total derivative on a tangent space field is defined by $\nabla_{\mu} Q^i \equiv D_{\mu} Q^i$.

field, its first and second order derivatives and $L_0 = L_0(\omega_\mu^{ef}, \partial_\nu\omega_\mu^{ef}, \partial_\rho\partial_\nu\omega_\mu^{ef})$ obeys local invariance under (3). This enables us to use the results presented elsewhere [21] to construct a gauge formulation for higher order gravitation theories.

3.1 The field strengths

According to [21], we can re-express

$$\begin{aligned}\delta L_0 &= \frac{1}{2}\frac{\partial L_0}{\partial\omega_\mu^{ef}}\delta\omega_\mu^{ef} + \frac{1}{2}\frac{\partial L_0}{\partial\left(\partial_\nu\omega_\mu^{ef}\right)}\delta\partial_\nu\omega_\mu^{ef} \\ &\quad + \frac{1}{2}\frac{\partial L_0}{\partial\left(\partial_\rho\partial_\nu\omega_\mu^{ef}\right)}\delta\partial_\rho\partial_\nu\omega_\mu^{ef} \\ &\equiv 0\,,\end{aligned}$$

splitting it into a set of four hierarchical equations after substituting (3) and claiming the independence of the parameters ε^{ab} and their derivatives. Three of these functional equations are used to conclude that

$$L_0 = L_0(F,G); \quad \frac{\partial L_0}{\partial\omega_\mu^{ab}} \equiv 0\,, \tag{12}$$

where

$$F_{\mu\nu}^{ab} = \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} - \eta_{cd}\omega_\mu^{ac}\omega_\nu^{db} + \eta_{cd}\omega_\nu^{ac}\omega_\mu^{db} \tag{13}$$

and

$$G_{\beta\rho\sigma}^{ab} = D_\beta F_{\rho\sigma}^{ab} = \partial_\beta F_{\rho\sigma}^{ab} - \eta_{fd}\omega_\beta^{af}F_{\rho\sigma}^{db} + \eta_{fd}\omega_\beta^{af}F_{\rho\sigma}^{bd}\,. \tag{14}$$

The remaining hierarchical equation put in terms of the gauge fields F and G,

$$\frac{\partial L_0}{\partial F_{\rho\sigma}^{ad}}f_{bc,gh}^{ad}F_{\rho\sigma}^{gh} + \frac{\partial L_0}{\partial G_{\beta\rho\sigma}^{ad}}f_{bc,gh}^{ad}G_{\beta\rho\sigma}^{gh} \equiv 0\,, \tag{15}$$

imposes restrictions upon the functional form eventually chosen for L_0. Substituting the structure constants, this condition can be explicitly written as

$$\begin{aligned}&\frac{\partial L_0}{\partial F_{\rho\sigma}^{ad}}\left[\eta_{cg}\delta_b^a - \eta_{bg}\delta_c^a\right]F_{\rho\sigma}^{gd} \\ &\quad + \frac{\partial L_0}{\partial G_{\beta\rho\sigma}^{ad}}\left[\eta_{cg}\delta_b^a - \eta_{bg}\delta_c^a\right]G_{\beta\rho\sigma}^{gd} \equiv 0\,.\end{aligned} \tag{16}$$

3.2 Geometrical variables

In this section we will show how to interpret all objects and conditions of the previous sections in terms of a geometrical point of view. From (10) we read

$$\omega_\sigma^{eg} = \eta^{gc}h_c^\alpha\left(\partial_\sigma h_\alpha^e - \Gamma_{\sigma\alpha}^\nu h_\nu^e\right)$$

and therefore the field strength F is written as

$$F_{\beta\sigma}^{eg} = \eta^{gc}h_c^\alpha h_\gamma^e\left[\partial_\sigma\Gamma_{\beta\alpha}^\gamma - \partial_\beta\Gamma_{\sigma\alpha}^\gamma + \Gamma_{\beta\alpha}^\nu\Gamma_{\sigma\nu}^\gamma - \Gamma_{\sigma\alpha}^\nu\Gamma_{\beta\nu}^\gamma\right],$$

where we recognize the expression of the Riemann tensor [26],

$$R_{\sigma\beta\alpha}^\gamma \equiv \partial_\sigma\Gamma_{\beta\alpha}^\gamma - \partial_\beta\Gamma_{\sigma\alpha}^\gamma + \Gamma_{\beta\alpha}^\nu\Gamma_{\sigma\nu}^\gamma - \Gamma_{\sigma\alpha}^\nu\Gamma_{\beta\nu}^\gamma\,,$$

i.e.,

$$F_{\beta\sigma}^{eg} = \eta^{gc}h_c^\alpha h_\gamma^e R_{\sigma\beta\alpha}^\gamma\,. \tag{17}$$

The easiest way to find the geometrical counterpart of G is to apply the geometrizing relations (6) and (8):

$$\begin{aligned}h_a^\mu h_b^\nu G_{\beta\rho\sigma}^{ab} &= h_a^\mu h_b^\nu D_\beta F_{\rho\sigma}^{ab} = \delta_\beta F_{\rho\sigma}^{\mu\nu}\,; \\ F_{\rho\sigma}^{\mu\nu} &= h_a^\mu h_b^\nu F_{\rho\sigma}^{ab}\end{aligned}$$

and use (17). We arrive at

$$G_{\beta\rho\sigma}^{ab} = h_\mu^a h_\nu^b g^{\nu\alpha}\delta_\beta R_{\sigma\rho\alpha}^\mu\,, \tag{18}$$

which is the most natural equation one would expect in view of the relation (14) between F and G.

By means of the geometrical descriptions (17) and (18), we are able to find

$$\begin{aligned}\frac{\partial L_0}{\partial F_{\rho\sigma}^{ad}} &= \frac{\partial L_0}{\partial R_{\lambda\beta\alpha}^\gamma}\frac{\partial R_{\lambda\beta\alpha}^\gamma}{\partial F_{\rho\sigma}^{ad}} = \frac{\partial L_0}{\partial R_{\sigma\rho\alpha}^\gamma}\eta_{bd}h_\alpha^b h_a^\gamma\,, \\ \frac{\partial L_0}{\partial G_{\beta\rho\sigma}^{ad}} &= \frac{\partial L_0^{(4)}}{\partial\left(\delta_\lambda R_{\gamma\nu\alpha}^\mu\right)}\frac{\partial\left(\delta_\lambda R_{\gamma\nu\alpha}^\mu\right)}{\partial G_{\beta\rho\sigma}^{ad}} = \frac{\partial L_0}{\partial\left(\delta_\beta R_{\sigma\rho\alpha}^\mu\right)}g_{\alpha\omega}h_a^\mu h_d^\omega\,.\end{aligned}$$

With these derivatives, the condition (15) for the gauge Lagrangian is put in the form

$$\begin{aligned}&\frac{\partial L_0}{\partial R_{\sigma\rho\beta}^\theta}\left[\delta_\nu^\theta g_{\gamma\lambda} - \delta_\gamma^\theta g_{\nu\lambda}\right]R_{\sigma\rho\beta}^\lambda \\ &\quad + \left[\frac{\partial L_0}{\partial\left(\delta_\beta R_{\sigma\rho\alpha}^\gamma\right)}g_{\nu\lambda} - \frac{\partial L_0}{\partial\left(\delta_\beta R_{\sigma\rho\alpha}^\nu\right)}g_{\gamma\lambda}\right]\delta_\beta R_{\sigma\rho\alpha}^\lambda \equiv 0\,.\end{aligned} \tag{19}$$

This is a fundamental restriction upon the Lagrangians tentatively proposed for the theory, and it is quite useful in order to choose a specific suitable invariant.

4 Quadratic Lagrangian counting

Our goal here is to determine all possible independent quadratic Lagrangians constructed with the field strength tensors F and G considering their various symmetries. By quadratic Lagrangians we mean invariants of the type FF or GG, but not mixed terms like FG (obviously with the proper contraction of indices). We will also compute the linear case of the Einstein–Hilbert Lagrangian.

4.1 First order invariants

The symmetries to be considered in the construction of the invariants of the type FF are those inherited from F. Thus we have skew-symmetry in each pair of indices:

$F^{ab}_{\mu\nu} = -F^{ba}_{\mu\nu}$ and $F^{ab}_{\mu\nu} = -F^{ab}_{\nu\mu}$. Besides these, there is another one, which is unveiled by the geometrical form of F, (17), namely

$$R^{\gamma}_{\sigma\beta\alpha} + R^{\gamma}_{\beta\alpha\sigma} + R^{\gamma}_{\alpha\sigma\beta} \equiv 0\,,$$

the familiar first Bianchi identity, met with in the context of the general relativity.

Once algebra and space-time indices can be transformed into each other by means of a tetrad, we will consider a compact representation for F:

$$F^{ab}_{\mu\nu} \to F^{ab}_{\mu\nu} h^{\mu}_{c} h^{\nu}_{d} \equiv (abcd)\,.$$

Since the Lagrangians are all of the form F^2 with all allowed orders of contractions, it is always possible to rename dummy indices in such a way that the first F will keep its indices in alphabetic order. In the table below follow all available permutations for the second F:

	Fix. a	Fix. b	Fix. c	Fix. d
cyclic	$(abcd)$	$(bacd)$	$(cabd)$	$(dabc)$
	$(acdb)$	$(bcda)$	$(cbda)$	$(dbca)$
	$(adbc)$	$(bdac)$	$(cdab)$	$(dcab)$
non-cyclic	$(abdc)$	$(badc)$	$(cadb)$	$(dacb)$
	$(acbd)$	$(bcad)$	$(cbad)$	$(dbac)$
	$(adcb)$	$(bdca)$	$(cdba)$	$(dcba)$

(20)

By means of a change in one pair of indices, one can see that the non-cyclic permutations are all proportional to the cyclic ones. Considering only the cyclic permutations and changing two pairs of indices, the table is reduced to

	Fix. a	Fix. b	Fix. c	Fix. d
cyclical	$(abcd)$	–	–	–
	$(acdb)$	$(bcda)$	–	–
	$(adbc)$	$(bdac)$	$(cdab)$	–

The skew-symmetries of the first F (which has been taken in alphabetic order) leads one to restrict once more the possible contractions to the three quadratic invariants:

$$\begin{aligned} I^F_1 &= (abcd)(abcd)\,,\\ I^F_2 &= (abcd)(acdb)\,,\\ I^F_3 &= (abcd)(cdab)\,. \end{aligned} \tag{21}$$

We now analyze the invariants constructed with one trace of F. The only non-null type of trace concerns those obtained by contracting one index of the first pair with one index of the second pair, in view of the skew-symmetry of this object. All possibilities are proportional to

$$\mathrm{Tr}F \to h^{\nu}_{c} F^{ca}_{\mu\nu} h^{\mu}_{b} \equiv (\cdot\, ab\, \cdot) \text{ or } (\circ\, ab\, \circ)\,.$$

The quadratic invariants are given by

$$\begin{aligned} I^{\mathrm{Tr}F}_1 &= (\cdot\, ab\, \cdot)(\circ\, ab\, \circ)\,,\\ I^{\mathrm{Tr}F}_2 &= (\cdot\, ab\, \cdot)(\circ\, ba\, \circ)\,. \end{aligned} \tag{22}$$

Still, one can construct a linear invariant taking a double trace of F:

$$I^{\mathrm{TrTr}F} = h^{\nu}_{c} F^{ca}_{\mu\nu} h^{\mu}_{a} \equiv (\cdot \circ \circ \cdot)\,.$$

4.2 Second order invariants

Let us introduce a notation similar to the one used in the case of F, i.e.,

$$G^{ab}_{\beta\rho\sigma} \to h^{\beta}_{c} h^{\rho}_{d} h^{\sigma}_{e} G^{ab}_{\beta\rho\sigma} \equiv [abcde]\,,$$

where we identify the following symmetries:

(i) antisymmetry by permutation of indices in the first pair and the last,

$$[abcde] = -[bacde] = -[abced]\,;$$

(ii) the Bianchi identity for the last three indices,

$$[abcde] + [abdec] + [abecd] = 0\,.$$

4.2.1 Invariants of GG kind

The quadratic combinations are now in a larger amount than in the F^2 case. In fact, we have five tables like (20), one to each letter labeling, since we can associate

$$[abcde] = c(abde)\,.$$

Using the symmetries cited above, one finds that the 5! G^2 invariants are reduced to just two kinds:

$$\begin{aligned} I^G_1 &= [abcde][abcde]\,,\\ I^G_2 &= [abcde][debac]\,. \end{aligned}$$

The detailed and cumbersome calculations are made in Appendix A.

4.2.2 Invariants involving traces

There are three independent types of traces for G:

$$\begin{aligned} T^{(1)}_{abc} &= h^{\beta}_{d} G^{da}_{\beta\rho\sigma} h^{\rho}_{b} h^{\sigma}_{c} \equiv [\cdot\, a \cdot bc]\,,\\ T^{(2)}_{abc} &= h^{\rho}_{d} G^{da}_{\beta\rho\sigma} h^{\beta}_{b} h^{\sigma}_{c} \equiv [\cdot\, ab \cdot c]\,,\\ T^{(3)}_{abc} &= g^{\beta\rho} G^{ab}_{\beta\rho\sigma} h^{\sigma}_{c} \equiv [ab \cdot \cdot\, c]\,. \end{aligned} \tag{23}$$

Again using symmetries (see Appendix A) we arrive at

$$\begin{aligned} \mathrm{Tr}\mathcal{G}_3 &= [\cdot\, ab \cdot c][\cdot\, a \cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{11} &= [\cdot\, ab \cdot c][\cdot\, bc \cdot a]\,,\\ \mathrm{Tr}\mathcal{G}_5 &= [\cdot\, ab \cdot c][\cdot\, c \cdot ab]\,, & \mathrm{Tr}\mathcal{G}_{14} &= [\cdot\, ab \cdot c][\cdot\, ab \cdot c]\,,\\ \mathrm{Tr}\mathcal{G}_6 &= [ab \cdot \cdot\, c][\cdot\, a \cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{17} &= [ab \cdot \cdot\, c][ab \cdot \cdot\, c]\,,\\ \mathrm{Tr}\mathcal{G}_{10} &= [\cdot\, ab \cdot c][\cdot\, ba \cdot c]\,, & \mathrm{Tr}\mathcal{G}_{18} &= [ab \cdot \cdot\, c][ac \cdot \cdot\, b]\,, \end{aligned} \tag{24}$$

while for double traces we have

$$\begin{aligned} \mathrm{TrTr}\mathcal{G}_1 &= [\cdot \circ b \cdot \circ][\cdot \circ b \cdot \circ]\,,\\ \mathrm{TrTr}\mathcal{G}_2 &= [\circ\, b \cdot \cdot \circ][\circ\, b \cdot \cdot \circ]\,,\\ \mathrm{TrTr}\mathcal{G}_3 &= [\cdot \circ b \cdot \circ][\circ\, b \cdot \cdot \circ]\,. \end{aligned} \tag{25}$$

4.3 Bianchi identities

As we already said, until now we have not used the first Bianchi identity:

$$R^{\sigma\chi\rho\beta} + R^{\chi\rho\sigma\beta} + R^{\rho\sigma\chi\beta} \equiv 0\,. \tag{26}$$

In geometrical variables, the cyclic property of G is translated to the second Bianchi identity:

$$\delta^\mu R^{\sigma\chi\rho\beta} + \delta^\sigma R^{\chi\mu\rho\beta} + \delta^\chi R^{\mu\sigma\rho\beta} \equiv 0\,.$$

These identities reduce the number of independent invariants, since $F \propto R$ and $G \propto \delta R$.

4.3.1 Reducing invariants

Let us begin by invariants of form F^2. The first three are (21)

$$\begin{aligned} I_1^F &= R_{\sigma\rho\chi\kappa} R^{\sigma\rho\chi\kappa}\,,\\ I_2^F &= R_{\sigma\rho\chi\kappa} R^{\sigma\chi\rho\kappa}\,,\\ I_3^F &= R_{\sigma\rho\chi\kappa} R^{\chi\kappa\sigma\rho}\,. \end{aligned}$$

As a consequence of the first Bianchi identity (26) and the skew-symmetries, the curvature tensor obeys

$$R_{\sigma\rho\chi\kappa} = R_{\chi\kappa\sigma\rho}\,. \tag{27}$$

Then

$$I_3^F = R_{\sigma\rho\chi\kappa} R^{\chi\kappa\sigma\rho} = R_{\sigma\rho\chi\kappa} R^{\sigma\rho\chi\kappa} = I_1^F\,,$$

while for I_2^F one finds

$$\begin{aligned} I_2^F &= R_{\sigma\rho\chi\kappa} R^{\sigma\chi\rho\kappa} = -(R_{\rho\chi\sigma\kappa} + R_{\chi\sigma\rho\kappa}) R^{\sigma\chi\rho\kappa}\\ &= -R_{\chi\rho\sigma\kappa} R^{\chi\sigma\rho\kappa} + R_{\chi\sigma\rho\kappa} R^{\chi\sigma\rho\kappa}\\ &= -I_2^F + I_1^F\,,\\ 2I_2^F &= I_1^F\,, \end{aligned}$$

which leaves us with only one invariant of this kind, I_1^F.

Now, we translate the trace-like invariants (22) in a geometrical form:

$$\begin{aligned} I_1^{\mathrm{Tr}F} &= R^\rho_{\rho\mu\nu} R^{\mu\nu\sigma}_\sigma\,,\\ I_2^{\mathrm{Tr}F} &= R^\rho_{\rho\mu\nu} R^{\nu\mu\sigma}_\sigma\,. \end{aligned}$$

Since the Ricci tensor $R_{\mu\nu} \equiv R^\rho_{\rho\mu\nu}$ is symmetric,[2] we have in fact only one invariant, $I_1^{\mathrm{Tr}F} = R_{\mu\nu}R^{\mu\nu}$.

At last, the only invariant of double traced form in F is

$$I^{\mathrm{TrTr}F} = R\,.$$

Analogously, in view of the Bianchi identities, only four invariants of the type G^2 remain (see Appendix A):

$$\begin{aligned} I_1^G &= \delta_\beta R_{\sigma\rho\chi\kappa} \delta^\beta R^{\sigma\rho\chi\kappa}\,, & \mathrm{TrTr}\mathcal{G}_2 &= \delta^\rho R_{\rho\chi} \delta_\mu R^{\mu\chi}\,,\\ \mathrm{Tr}\mathcal{G}_{10} &= \delta_\beta R_{\sigma\chi} \delta^\chi R^{\sigma\beta}\,, & \mathrm{Tr}\mathcal{G}_{14} &= \delta_\beta R_{\sigma\chi} \delta^\beta R^{\sigma\chi}\,. \end{aligned}$$

[2] This is a consequence of the first Bianchi identity.

5 Gauge invariance condition

With the invariants constructed above we collect seven types of Lagrangians for the gravitational field:

Lagr.	Inv.	Gauge Form	Geom. Form
$L_0^{(R_1)}$	$(I^{\mathrm{TrTr}F})^n$	$\left(F^{ab}_{ba}\right)^n$	$R^n,\ n=1,2$
$L_0^{(R_2)}$	$I_1^{\mathrm{Tr}F}$	$F^a_{b\mu a} F^{b\mu c}_c$	$R_{\mu\nu}R^{\mu\nu}$
$L_0^{(R_3)}$	I_1^F	$F^{ab}_{\mu\nu} F^{\mu\nu}_{ab}$	$R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$
$L_0^{(G_1)}$	$\mathrm{TrTr}\mathcal{G}_2$	$G^{\beta a}_{ab\beta} G^{cb\mu}_{\mu c}$	$\delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi}$
$L_0^{(G_2)}$	$\mathrm{Tr}\mathcal{G}_{14}$	$G^{ab}_{\mu a\sigma} G^{\mu c\sigma}_{cb}$	$\delta_\beta R_{\sigma\chi}\delta^\beta R^{\sigma\chi}$
$L_0^{(G_3)}$	$\mathrm{Tr}\mathcal{G}_{10}$	$G^{abe}_{a\sigma} G^{d\sigma}_{deb}$	$\delta_\beta R_{\sigma\chi}\delta^\chi R^{\sigma\beta}$
$L_0^{(G_4)}$	I_1^G	$G^{ab}_{\mu\nu\lambda} G^{\mu\nu\lambda}_{ab}$	$\delta_\beta R_{\sigma\rho\chi\kappa}\delta^\beta R^{\sigma\rho\chi\kappa}$

(28)

We are considering Lagrangians only up to quadratic order in F and/or G, which also includes the linear invariant $I^{\mathrm{TrTr}F} = R$ and the square R^2. Actually, one can observe that if any invariant fulfills the gauge invariance condition, then any of its powers will, since this condition is linear in the derivatives $\partial L_0/\partial F$ and $\partial L_0/\partial G$. For instance,

$$L_0 = I^n\,, \quad \frac{\partial L_0}{\partial F} = nI^{n-1}\frac{\partial I}{\partial F}\,.$$

Therefore,

$$\frac{\partial I}{\partial F}[\ldots]F = 0 \Rightarrow \frac{\partial L_0}{\partial F}[\ldots]F = 0\,,$$

and the same follows for G.

Using the skew-symmetry $\nu \leftrightarrow \gamma$ of (19) and the symmetry properties of the Riemann tensor, one can easily verify that all Lagrangian densities listed in (28) accomplish the gauge invariance condition. Then, any function of these invariants expressible in a Taylor series also will fulfill the gauge invariance condition.

6 Equations of motion

Here we will concentrate our attention on the effect of the term

$$L_0^{(G_1)} = \frac{1}{2} h h^a_\sigma h^\nu_c G^{\beta\sigma}_{ab\beta} G^{cb\mu}_{\mu\nu} = \frac{1}{8} h \delta^\rho R \delta_\rho R$$

on a gravitational theory based on the Einstein–Hilbert action plus the $L_0^{(G_1)}$ term. This Lagrangian density is equivalent, by the Bianchi identity, to the form $L_{\mathrm{P}} = \frac{1}{2} h \delta^\rho R_{\rho\chi} \delta_\mu R^{\mu\chi}$, which is clearly analogous to Podolsky's second order term for electrodynamics ($L_{\mathrm{Podolsky}} \propto \partial^\rho F_{\rho\chi} \partial_\mu F^{\mu\chi}$). The choice of the particular Lagrangian $L_0^{(G_1)}$ is mainly motivated by this analogy. Besides this, the $L_0^{(G_1)}$ term also can be viewed as a kind of kinetic term for the scalar curvature, which approximates such a description to the usual scalar fields. Moreover, this scalar is, up to a surface term, present in the Schwinger–DeWitt renormalized effective action for a scalar field on a curved background [27]. Therefore, the field theory constructed on the basis on $L_0^{(G_1)}$ can be considered an effective gravitational theory.

Taking a functional variation of the tetrad field, one finds

$$h = \sqrt{-g}\,, \quad \delta h = \frac{1}{2} h g^{\lambda\nu} \delta g_{\lambda\nu} = h g^{\lambda\nu} h^a_\lambda \eta_{ab} \delta h^b_\nu$$

and

$$\delta L_0^{(G_1)} = \frac{1}{4}\partial_\rho (h\partial^\rho R \delta R) - \frac{1}{4}\delta R \partial_\rho (h \partial^\rho R) + \frac{1}{4} h \left[\frac{1}{2} g^{\lambda\nu} \partial^\rho R \partial_\rho R - g^{\mu\nu} g^{\rho\lambda} \partial_\mu R \partial_\rho R\right] h^a_\lambda \eta_{ab} \delta h^b_\nu \,.$$

On calculating the equations of motion, we must give special attention to the last term, involving

$$\delta R = -2 R_{\mu\beta} g^{\mu\nu} g^{\beta\lambda} h^a_\lambda \eta_{ab} \delta h^b_\nu + \frac{1}{h}\partial_\alpha \left[h \left(g^{\mu\nu} \delta \Gamma^\alpha_{\mu\nu} - g^{\nu\alpha} \delta \Gamma^\beta_{\nu\beta}\right)\right] ,$$

which will include several integrations by parts. After these integrations and some cumbersome calculations, one finds

$$\delta L_0^{(G_1)} = \frac{1}{2}\partial_\theta \mathcal{V}^\theta + \frac{1}{2} h \left[\delta_\lambda \delta_\nu [\Diamond R] + \frac{1}{2}\delta_\lambda R \delta_\nu R - R_{\lambda\nu} \Diamond R - g_{\lambda\nu} \Diamond [\Diamond R] - \frac{1}{4} g_{\lambda\nu} \delta^\rho R \delta_\rho R\right] h^\lambda_a \eta^{ab} \delta h^\nu_b \,,$$

where

$$\mathcal{V}^\theta \equiv -\frac{1}{2}\left(g^{\mu\nu}\delta\Gamma^\theta_{\mu\nu} - g^{\nu\theta}\delta\Gamma^\beta_{\nu\beta}\right)\partial_\rho (h \partial^\rho R) - \frac{1}{2} h \partial^\theta R \delta R + \frac{1}{4} h (g^{\mu\nu} g^{\alpha\beta} - g^{\nu\alpha} g^{\mu\beta}) \delta_\alpha [\Diamond R] \times \left(\delta^\theta_\nu \delta^\lambda_\mu \delta^\eta_\beta + \delta^\theta_\mu \delta^\lambda_\beta \delta^\eta_\nu - \delta^\theta_\beta \delta^\lambda_\nu \delta^\eta_\mu\right) \delta g_{\lambda\eta} \,,$$

and

$$\Diamond \equiv \delta_\beta \delta^\beta$$

is the Laplace–Beltrami operator on the Riemannian space.

Therefore, the second order contribution to the equation of motion will be

$$H^b_\nu \equiv h^{b\lambda} \delta_\lambda \delta_\nu [\Diamond R] + \frac{1}{2} h^{b\lambda} \delta_\lambda R \delta_\nu R - R^b_\nu \Diamond R - h^b_\nu \delta_\beta \delta^\beta [\Diamond R] - \frac{1}{4} h^b_\nu \delta^\rho R \delta_\rho R \,. \tag{29}$$

Furthermore, if we include the usual first order Einstein–Hilbert and matter Lagrangian densities,

$$S_T = \int \mathrm{d}^n x \left(-\frac{hR}{2\chi} - \frac{\beta}{\chi} L_0^{(G_1)} + h \mathcal{L}_{\text{matter}} \right) ,$$

the field equations become

$$G^b_\nu + \beta H^b_\nu = \chi T^b_\nu \,, \tag{30}$$

or, in geometrical form,

$$R_{\lambda\nu} - \frac{1}{2} g_{\lambda\nu} R + \beta \left[\delta_\lambda \delta_\nu (\Diamond R) + \frac{1}{2}\delta_\lambda R \delta_\nu R - R_{\lambda\nu} \Diamond R - g_{\lambda\nu} \Diamond (\Diamond R) - \frac{1}{4} g_{\lambda\nu} \delta^\rho R \delta_\rho R\right] = \chi T_{\lambda\nu} \,,$$

where G^b_ν is the Einstein tensor and

$$T_{\lambda\nu} \equiv \frac{2}{h}\frac{\delta (h \mathcal{L}_{\text{matter}})}{\delta g^{\lambda\nu}}$$

is the energy-momentum tensor of the matter fields written in terms of the metric field.

By analogy to the Alekseev–Arbuzov–Baikov case [22], one could expect that the higher order terms, which can be up to sixth derivative order, would be related to infrared corrections to general relativity, giving observable physical effects at large scales.

6.1 Covariant conservation of $T_{\lambda\nu}$

Taking the covariant divergence of (30), we have

$$\delta^\nu G_{\nu\alpha} + \beta \delta^\nu H_{\nu\alpha} = \chi \delta^\nu T_{\nu\alpha} \,.$$

Now, from the first order case, we know that

$$\delta^\nu G_{\nu\alpha} \equiv 0 \,.$$

Applying the divergence to (29), one finds

$$\delta^\nu H_{\nu\alpha} = \delta^\nu \delta_\nu \delta_\alpha \Diamond R - g_{\nu\alpha} \delta^\nu \Diamond [\Diamond R] + \frac{1}{2}\delta_\nu R \delta^\nu \delta_\alpha R + \frac{1}{2}\delta^\nu \delta_\nu R \delta_\alpha R - \delta^\nu R_{\nu\alpha} \Diamond R - R_{\nu\alpha} \delta^\nu \Diamond R - \frac{1}{4} g_{\nu\alpha} \delta^\nu \left(\delta^\rho R \delta_\rho R\right) .$$

Using the commutation relation

$$[\delta_\nu, \delta_\alpha] A^\tau = R^{\tau\xi}_{\alpha\nu} A_\xi$$

and the second Bianchi identity, we arrive at

$$\delta^\nu H_{\nu\alpha} = R_{\alpha\xi} \delta^\xi \Diamond R - R_{\nu\alpha} \delta^\nu \Diamond R + \frac{1}{2}\delta_\nu R \delta^\nu \delta_\alpha R - \frac{1}{2}\delta^\rho R \delta_\alpha \delta_\rho R = 0 \,.$$

Then the covariant conservation of $T_{\mu\nu}$ is established:

$$\delta^\mu (G_{\mu\nu} + \beta H_{\mu\nu}) \equiv 0 \Longrightarrow \delta^\mu T_{\mu\nu} = 0 \,,$$

as expected from the coordinate invariance of the Lagrangian density.

6.2 Static isotropic solution

In the case of a static isotropic metric,

$$\mathrm{d}s^2 = \mathrm{e}^{\nu(r)} \mathrm{d}t^2 - \mathrm{e}^{\lambda(r)} \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \, \mathrm{d}\phi^2$$

in the vacuum, the equations of motion (30) are reduced, in the linear approximation, to the following coupled linear equations:

$$\nu''+\frac{2}{r}\nu'+\beta\Big(\nu^{(6)}+\frac{6}{r}\nu^{(5)}-\frac{2}{r}\lambda^{(5)}-\frac{2}{r^2}\lambda^{(4)} +\frac{8}{r^3}\lambda'''-\frac{24}{r^4}\lambda''+\frac{48}{r^5}\lambda'-\frac{48}{r^6}\lambda\Big)=0\,,$$

$$\frac{1}{2}\Big(\frac{\lambda'}{r}-2\frac{\lambda}{r^2}+\frac{\nu'}{r}-\nu''\Big) -\beta\Big(\nu^{(6)}+\frac{3}{r}\nu^{(5)}-\frac{12}{r^2}\nu^{(4)}+12\frac{\nu'''}{r^3}-\frac{2}{r}\lambda^{(5)} +\frac{4}{r^2}\lambda^{(4)}+8\frac{\lambda'''}{r^3}-48\frac{\lambda''}{r^4}+96\frac{\lambda'}{r^5}-96\frac{\lambda}{r^6}\Big)=0\,.$$

To solve this system, we use the Frobenius method, based on a series expansion:

$$\nu(r)=\sum_n \nu_n r^{s+n}\,,\quad \lambda(r)=\sum_n \lambda_n r^{s+n}\,.$$

From the first terms in the series, we find $s=-1$, and the recursion relations above become

$$\lambda_{n+4}=\frac{\nu_n(n-2)}{4\beta(n+4)(n+2)(n+1)\left(n-\frac{1}{2}\right)}\,,$$
$$\nu_{n+4}=-\frac{\nu_n}{2\beta(n+4)(n+3)(n+2)\left(n-\frac{1}{2}\right)}\,,$$

so that the solution can be written as

$$\nu(r)=\sum_{m=0}^{3}\nu_m r^{m-1}\Big(1+\sum_{n=0}^{\infty}c_{nm}\Big)\,,$$
$$\lambda(r)=\sum_{m=0}^{3}\lambda_m r^{m-1} -\sum_{m=0}^{3}\nu_m r^{m-1}\sum_{n=0}^{\infty}\frac{(4n+m-2)(4n+3+m)}{2(4n+1+m)}c_{nm}\,,$$

where ν_m and λ_m, with $m\in\{0,1,2,3\}$, are the integration constants specified by the boundary conditions, and

$$c_{nm}\equiv\Big(-\frac{r^4}{2\beta}\Big)^{n+1}\frac{(m+1)!}{(4n+m+4)!} \times\frac{(4n+m+1)!!!!}{(m+1)!!!!}\,\frac{\left(m-\frac{9}{2}\right)!!!!}{\left(4n+m-\frac{1}{2}\right)!!!!}$$

The notation $a!!!!$ stands for

$$(a+4)!!!!=(a+4)\cdot a!!!!\,.$$

The convergence of the series, tested by the ratio test,

$$\lim_{n\to\infty}\left|\frac{\nu_{n+1}}{\nu_n}\right|=\left|\frac{r^4}{2\beta}\right|\lim_{n\to\infty}\left|D_{n,m}\frac{1}{(4n+m+7)}\right|=0\,,$$
$$\lim_{n\to\infty}\left|\frac{\lambda_{n+1}}{\lambda_n}\right| =\left|\frac{r^4}{2\beta}\right|\lim_{n\to\infty}\left|D_{nm}\right| \times\lim_{n\to\infty}\left|\frac{(4n+m+2)(4n+m+1)}{(4n+m+5)(4n+m+3)(4n+m-2)}\right| =0\,,$$
$$D_{nm}\equiv\frac{1}{(4n+m+8)(4n+m+6)\left(4n+m+\frac{7}{2}\right)}\,,$$

shows that both are convergent with an infinite radius of convergence.

Therefore, in the first order approximation for β, we have

$$\nu(r)=\frac{\nu_0}{r}\Big(1+\frac{1}{24\beta}r^4\Big)+\nu_1\Big(1-\frac{1}{60\beta}r^4\Big) +\nu_2 r\Big(1-\frac{1}{360\beta}r^4\Big)+\nu_3 r^2\Big(1-\frac{1}{1050\beta}r^4\Big)+\mathcal{O}(\beta^2)$$
$$\lambda(r)=-\frac{\nu_0}{r}+\lambda_1+\lambda_2 r+\lambda_3 r^2 +\frac{1}{6\beta}\Big(-\frac{\nu_0}{4}r^3+\frac{\nu_1}{10}r^4+\frac{\nu_2}{60}r^5+\frac{\nu_3}{175}r^6\Big)+\mathcal{O}(\beta^2)\,. \tag{31}$$

An analysis of the solution (31) reveals the expected weak field behavior at short scales, ν_0/r, and the deviation from this for the meso-scale, since we are dealing only with the linear approximation. Correspondingly, we find $\nu_0=2GM/c^2$ where M is the mass of the central body.

The remaining integration constants set scale distances where modifications of the Newtonian behavior appear. For instance, consider the Einstein–Hilbert theory with cosmological constant. The static spherically symmetric solution is

$$\nu(r)=-\lambda(r)=1-\frac{2GM}{c^2}\frac{1}{r}-\frac{\Lambda}{3}r^2\,,$$

where the cosmological constant sets a scale distance given by the de Sitter pseudo-radius.

Analogously, in our case, the ν_1 constant sets a constant potential, which can be a mean nonlocal value of the effective Lagrangian proposed, ν_2 sets a scale distance where a constant mean force appears, and ν_3 represents a gradient of force, in the same way as the cosmological constant in the example above. A similar reasoning can be developed for the other constants in the model.

The contribution of each constant to the net force could be fixed by requiring that it fits the observational data for the tests of the gravitation. This task deserves a careful investigation of its own and is presently under investigation by the authors by means of the study of galaxy rotation curves, geodesic motion, perihelion shift, gravitational lenses and redshift.

7 Conclusion

We have applied second order gauge theory [21] to local gauge theory for the homogeneous Poincaré group. It was found that the geometrical counterparts of the usual field strength F and of the second order field strength $G = DF$ are the Riemann tensor R and its (space-time) covariant derivative δR. There followed an analysis of the second order invariants composed with geometrical entities.

We demonstrate – employing the symmetry properties of the curvature tensor – that the only independent Lagrangian densities for the gravitational field in a Riemannian manifold of arbitrary dimension are the seven ones listed in (28). Linear combinations of terms proportional to powers of R, as the familiar quadratic term in the curvature, are of first order in the gauge potential ω; therefore, in the context of the second order gauge theory, the contributions of second order in the Lagrangian density, which are those including second derivatives of the gauge potential, are of type δR.

We derived equations of motion using a particularly simple choice for the second order gauge Lagrangian inspired in Podolsky's proposal for generalized electrodynamics. We found the static isotropic solution of these equations in the linear approximation, showing that at short distances the gravitational field behaves exactly as Newton's law, but at meso-large distance scales the higher order contribution dominates, exhibiting a modified potential.

In the future, we will study other solutions of these field equations, searching for massive modes that do not violate local gauge symmetry. Our guide in these calculations shall be the treatment given in [21] to the U(1) case, where an effective mass for the photon was derived. To do this, one naturally must concern oneself with the determination of the conserved current associated with the local Lorentz symmetry and the relationship to the global diffeomorphic invariance of the theory.

Another perspective is to apply the second order equations of motion (30) to a Friedmann–Robertson–Walker metric. The goal is to seek for accelerated regimes of the cosmological model arising from the higher order terms. This proposal is now under investigation.

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Appendix: Counting second order invariants

A.1 Counting GG invariants

First, let us analyze how many possible contractions there are of the kind GG. This is done by means of tables as in Sect. 4.1. The first one is constructed fixing, for instance, the last index:

Fix. e	Fix. a	Fix. b	Fix. c	Fix. d
	$(abcd)$	$(bacd)$	$(cabd)$	$(dabc)$
cyclic	$(acdb)$	$(bcda)$	$(cbda)$	$(dbca)$
	$(adbc)$	$(bdac)$	$(cdab)$	$(dcab)$
	$(abdc)$	$(badc)$	$(cadb)$	$(dacb)$
non-cycl.	$(acbd)$	$(bcad)$	$(cbad)$	$(dbac)$
	$(adcb)$	$(bdca)$	$(cdba)$	$(dcba)$

Analogous tables result when we fix the indices d, c, b and a. For each table, non-cyclic permutations are equivalent to cyclic ones, giving:

Fix. e	Fix. a	Fix. b	Fix. c	Fix. d
	$(abcd)$	–	–	–
cyclic	$(acdb)$	$(bcda)$	–	–
	$(adbc)$	$(bdac)$	$(cdab)$	–

and similarly for the other four tables.

Using the cyclic permutation symmetry, one can identify elements of different tables, reducing the number of invariants. By the skew-symmetry in the first G, and renaming dummy indices, there follows

$$\begin{aligned} \mathcal{G}_1 &= [abcde][abcde]\,, & \mathcal{G}_6 &= [abcde][cdbea]\,,\\ \mathcal{G}_2 &= [abcde][beacd]\,, & \mathcal{G}_7 &= [abcde][adbec]\,,\\ \mathcal{G}_3 &= [abcde][adceb]\,, & \mathcal{G}_8 &= [abcde][acbde]\,,\\ \mathcal{G}_4 &= [abcde][aecbd]\,, & \mathcal{G}_9 &= [abcde][acdeb]\,,\\ \mathcal{G}_5 &= [abcde][debac]\,, & \mathcal{G}_{10} &= [abcde][abdce]\,. \end{aligned}$$

One can further apply the cyclic permutation symmetry to the first G in these remaining invariants and reduce even more the number of independent quantities. Beginning with $\mathcal{G}_{10}$:

$$\mathcal{G}_{10} = -([abdec] + [abecd])[abdce] = \mathcal{G}_1 - \mathcal{G}_{10} \Rightarrow 2\mathcal{G}_{10} = \mathcal{G}_1\,.$$

On the other hand, for $\mathcal{G}_9$:

$$\mathcal{G}_9 = -([abdec] + [abecd])[acdeb] = 2\mathcal{G}_4 + \mathcal{G}_9 \Rightarrow \mathcal{G}_4 = 0\,.$$

Proceeding in the same way, one finds the following identities:

$$\begin{gathered} 2\mathcal{G}_{10} = \mathcal{G}_1\,; \quad 2\mathcal{G}_6 = \mathcal{G}_5\,;\\ \mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}_4 = \mathcal{G}_7 = \mathcal{G}_8 = \mathcal{G}_9 = 0\,. \end{gathered}$$

which give two independent invariants,

$$I_1^G = [abcde][abcde]\,, \quad I_2^G = [abcde][debac]\,.$$

A.2 Counting $(\mathrm{Tr}G)^2$ invariants

Starting with the three independent traces listed in (23), and considering the skew-symmetries, the possible quadratic combinations are:

$$\begin{aligned}
\mathrm{Tr}\mathcal{G}_1 &= [\cdot a\cdot bc][\cdot a\cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{10} &= [\cdot ab\cdot c][\cdot ba\cdot c]\,,\\
\mathrm{Tr}\mathcal{G}_2 &= [\cdot a\cdot bc][\cdot b\cdot ac]\,, & \mathrm{Tr}\mathcal{G}_{11} &= [\cdot ab\cdot c][\cdot bc\cdot a]\,,\\
\mathrm{Tr}\mathcal{G}_3 &= [\cdot ab\cdot c][\cdot a\cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{12} &= [\cdot ab\cdot c][\cdot ca\cdot b]\,,\\
\mathrm{Tr}\mathcal{G}_4 &= [\cdot ab\cdot c][\cdot b\cdot ac]\,, & \mathrm{Tr}\mathcal{G}_{13} &= [\cdot ab\cdot c][\cdot cb\cdot a]\,,\\
\mathrm{Tr}\mathcal{G}_5 &= [\cdot ab\cdot c][\cdot c\cdot ab]\,, & \mathrm{Tr}\mathcal{G}_{14} &= [\cdot ab\cdot c][ab\cdot\cdot c]\,,\\
\mathrm{Tr}\mathcal{G}_6 &= [ab\cdot\cdot c][\cdot a\cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{15} &= [\cdot ab\cdot c][ac\cdot\cdot b]\,,\\
\mathrm{Tr}\mathcal{G}_7 &= [ab\cdot\cdot c][\cdot c\cdot ab]\,, & \mathrm{Tr}\mathcal{G}_{16} &= [\cdot ab\cdot c][bc\cdot\cdot a]\,,\\
\mathrm{Tr}\mathcal{G}_8 &= [\cdot ab\cdot c][\cdot ab\cdot c]\,, & \mathrm{Tr}\mathcal{G}_{17} &= [ab\cdot\cdot c][ab\cdot\cdot c]\,,\\
\mathrm{Tr}\mathcal{G}_9 &= [\cdot ab\cdot c][\cdot ac\cdot b]\,, & \mathrm{Tr}\mathcal{G}_{18} &= [ab\cdot\cdot c][ac\cdot\cdot b]\,.
\end{aligned}$$

The last two invariants cannot be converted into any other ones using the symmetries at our disposal. Each one of the preceding $\mathrm{Tr}\mathcal{G}$ must be analyzed, case by case, in a search for an eventual interdependence.

Take, for example, the 16th term, and rewrite it as

$$\begin{aligned}
&\mathrm{Tr}\mathcal{G}_{16} = -[\cdot acb\cdot][bc\cdot\cdot a] - [\cdot a\cdot cb][bc\cdot\cdot a]\\
&\Rightarrow 2\mathrm{Tr}\mathcal{G}_{16} = \mathrm{Tr}\mathcal{G}_7\,.
\end{aligned}$$

Repeat the reasoning for, say, the 15th invariant:

$$\begin{aligned}
\mathrm{Tr}\mathcal{G}_{15} &= -[\cdot acb\cdot][ac\cdot\cdot b] - [\cdot a\cdot cb][ac\cdot\cdot b]\\
&= \mathrm{Tr}\mathcal{G}_{14} - \mathrm{Tr}\mathcal{G}_6\,.
\end{aligned}$$

As soon as we perform this same check for all the above invariants, only eight of them are kept:

$$\begin{aligned}
\mathrm{Tr}\mathcal{G}_3 &= [\cdot ab\cdot c][\cdot a\cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{11} &= [\cdot ab\cdot c][\cdot bc\cdot a]\,,\\
\mathrm{Tr}\mathcal{G}_5 &= [\cdot ab\cdot c][\cdot c\cdot ab]\,, & \mathrm{Tr}\mathcal{G}_{14} &= [\cdot ab\cdot c][\cdot ab\cdot c]\,,\\
\mathrm{Tr}\mathcal{G}_6 &= [ab\cdot\cdot c][\cdot a\cdot bc]\,, & \mathrm{Tr}\mathcal{G}_{17} &= [ab\cdot\cdot c][ab\cdot\cdot c]\,,\\
\mathrm{Tr}\mathcal{G}_{10} &= [\cdot ab\cdot c][\cdot ba\cdot c]\,, & \mathrm{Tr}\mathcal{G}_{18} &= [ab\cdot\cdot c][ac\cdot\cdot b]\,.
\end{aligned} \tag{A.1}$$

A.3 Counting $(\mathrm{TrTr}G)^2$ invariants

From $T^{(1)}_{abc} \equiv [\cdot a\cdot bc]$ one can take a trace again:

$$T^{(1)}_c \equiv [\cdot\circ\cdot\circ c]\,.$$

From $T^{(2)}_{abc} \equiv [\cdot ab\cdot c]$ one finds $T_c \equiv [\cdot\circ\circ\cdot c]$, which can be reduced to $T^{(1)}_c$ using the G skew-symmetry in the first two indexes and changing dummy indexes. Another possible trace is constructed from $T^{(2)}_{abc}$:

$$T^{(2)}_b \equiv [\cdot\circ b\cdot\circ]\,. \tag{A.2}$$

But it also is not independent of $T^{(1)}_c$:

$$\begin{aligned}
T^{(2)}_b &\equiv [\cdot\circ b\cdot\circ] = -[\cdot\circ\cdot\circ b] - [\cdot\circ\circ b\cdot]\\
&= -T^{(1)}_b - [\circ\cdot\circ\cdot b] = -2T^{(1)}_b\,.
\end{aligned}$$

Let us set $T^{(2)}_b$ as the independent double trace.

There is an internal double trace of $T^{(3)}_{abc} \equiv [ab\cdot\cdot c]$, which is independent of $T^{(2)}_c$:

$$T^{(3)}_b \equiv [\circ b\cdot\cdot\circ]\,. \tag{A.3}$$

The other double trace of $T^{(3)}_{abc}$ is

$$T_b \equiv [b\circ\cdot\cdot\circ] = -T^{(3)}_b\,.$$

Then we have the following set of independent double traces:

$$\begin{aligned}
\mathrm{TrTr}\mathcal{G}_1 &= [\cdot\circ b\cdot\circ][\cdot\circ b\cdot\circ]\,,\\
\mathrm{TrTr}\mathcal{G}_2 &= [\circ b\cdot\cdot\circ][\circ b\cdot\cdot\circ]\,,\\
\mathrm{TrTr}\mathcal{G}_3 &= [\cdot\circ b\cdot\circ][\circ b\cdot\cdot\circ]\,.
\end{aligned} \tag{A.4}$$

A.4 Reducing the G^2 invariants using Bianchi identities

Consider the reduction of the number of quadratic invariants in G by means of the Bianchi identities. Using the geometric form, the first two invariants are

$$I^G_1 = \delta_\beta R_{\sigma\rho\chi\kappa}\delta^\beta R^{\sigma\rho\chi\kappa}\,, \quad I^G_2 = \delta_\beta R_{\sigma\rho\chi\kappa}\delta^\chi R^{\beta\kappa\sigma\rho}\,.$$

Applying the second Bianchi identity to I^G_2 we have

$$\begin{aligned}
&I^G_2 = -\delta_\beta R_{\sigma\rho\chi\kappa}(\delta^\chi R^{\kappa\beta\rho\sigma} + \delta^\beta R^{\chi\kappa\rho\sigma}) = I^G_1 - I^G_2\\
&\Rightarrow 2I^G_2 = I^G_1\,,
\end{aligned}$$

therefore, it is sufficient to consider only I^G_1.

Let us analyze now the trace invariants in G, (24):

$$\begin{aligned}
\mathrm{Tr}\mathcal{G}_3 &= \delta_\beta R^\zeta_{\sigma\zeta\chi}\delta_\mu R^{\sigma\beta\chi\mu}\,, & \mathrm{Tr}\mathcal{G}_{11} &= \delta_\beta R_{\sigma\chi}\delta^\sigma R^{\chi\beta}\,,\\
\mathrm{Tr}\mathcal{G}_5 &= \delta_\beta R^\rho_{\sigma\rho\chi}\delta_\mu R^{\beta\chi\sigma\mu}\,, & \mathrm{Tr}\mathcal{G}_{14} &= \delta_\beta R_{\sigma\chi}\delta^\beta R^{\sigma\chi}\,,\\
\mathrm{Tr}\mathcal{G}_6 &= \delta^\rho R_{\sigma\rho\chi\zeta}\delta_\kappa R^{\sigma\chi\zeta\kappa}\,, & \mathrm{Tr}\mathcal{G}_{17} &= \delta^\rho R_{\sigma\rho\chi\kappa}\delta_\mu R^{\sigma\mu\chi\kappa}\,,\\
\mathrm{Tr}\mathcal{G}_{10} &= \delta_\beta R_{\sigma\chi}\delta^\chi R^{\sigma\beta}\,, & \mathrm{Tr}\mathcal{G}_{18} &= \delta^\rho R_{\sigma\rho\chi\kappa}\delta_\mu R^{\chi\mu\sigma\kappa}\,.
\end{aligned}$$

Comparing $\mathrm{Tr}\mathcal{G}_{10}$ with $\mathrm{Tr}\mathcal{G}_{11}$ one sees that these two are the same invariant, due to the symmetry of Ricci tensor.

Using the second Bianchi identity, it follows that

$$\mathrm{Tr}\mathcal{G}_3 = \delta_\beta R_{\sigma\chi}g_{\mu\rho}(\delta^\beta R^{\rho\sigma\chi\mu} + \delta^\sigma R^{\beta\rho\chi\mu}) = \mathrm{Tr}\mathcal{G}_{14} - \mathrm{Tr}\mathcal{G}_{10}\,,$$

and in the same way

$$\mathrm{Tr}\mathcal{G}_5 = \mathrm{Tr}\mathcal{G}_6 = -\frac{1}{2}\mathrm{Tr}\mathcal{G}_{17} = -\mathrm{Tr}\mathcal{G}_{18} = -\mathrm{Tr}\mathcal{G}_3\,.$$

This shows that only $\mathrm{Tr}\mathcal{G}_{14}$ and $\mathrm{Tr}\mathcal{G}_{10}$ can hold independently.

We apply the same technique to the double traced invariants (25):

$$\begin{aligned}
\mathrm{TrTr}\mathcal{G}_1 &= \delta_\beta R\delta^\beta R\,,\\
\mathrm{TrTr}\mathcal{G}_2 &= \delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi}\,,\\
\mathrm{TrTr}\mathcal{G}_3 &= -\delta_\beta R\delta_\mu R^{\mu\beta}\,.
\end{aligned}$$

The second Bianchi identity shows us that we have only one invariant in such a case:

$$\begin{aligned}\mathrm{TrTr}\mathcal{G}_3 &= \delta^{\rho}_{\zeta} g^{\sigma\chi}\left(\delta_{\sigma} R^{\zeta}_{\beta\rho\chi} + \delta_{\rho} R^{\zeta}_{\sigma\beta\chi}\right)\delta_{\mu} R^{\mu\beta} \\ &= -2\mathrm{TrTr}\mathcal{G}_2 = -\frac{1}{2}\mathrm{TrTr}\mathcal{G}_1\,.\end{aligned}$$

References

1. P. Astier et al., Astron. Astrophys. **447**, 31 (2006)
2. A.G. Riess et al., Astrophys. J. **607**, 665 (2004)
3. S. Nojiri, S.D. Odintsov, hep-th/0608008
4. S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002)
5. S.M. Carroll, V. Duvvuri, M. Trodden, S. Turner, Phys. Rev. D **70**, 043 528 (2004)
6. U. Gunther, A. Zhuk, V. Bezerra, C. Romero, Class. Quantum Grav. **22**, 313 (2005)
7. N.D. Birrell, P.C.W. Davies, Quantum Fields in Curved Space (Cambridge University Press, New York, 1984)
8. R. Casana, C.A.M. de Melo, B.M. Pimentel, Proc. Sci. WC **2004**, 013 (2004)
9. R. Casana, C.A.M. de Melo, B.M. Pimentel, Brazil. J. Phys. **35**, 1151 (2005)
10. R. Casana, C.A.M. de Melo, B.M. Pimentel, Astrophys. Space Sci. **305**, 125 (2006)
11. B. Podolsky, Phys. Rev. **62**, 68 (1942)
12. B. Podolsky, C. Kikuchi, Phys. Rev. **65**, 228 (1944)
13. B. Podolsky, P. Schwed, Rev. Mod. Phys. **20**, 40 (1948)
14. K.S. Stelle, Gen. Relat. Grav. **9**, 353 (1978)
15. A. Accioly, A. Azeredo, H. Mukai, J. Math. Phys. **43**, 473 (2002)
16. R.P. Feynman, Feynman Lectures on Gravitation (Westview Press, Boulder, 2002)
17. M.H. Goroff, A. Sagnotti, Phys. Lett. B **160**, 81 (1985)
18. B.S. DeWitt, Dynamical Theory of Groups and Fields (Gordon & Breach Science Pub., New York, 1965)
19. S.M. Christensen, Phys. Rev. D **14**, 2490 (1976)
20. R. Utiyama, Phys. Rev. **101**, 1597 (1956)
21. R.R. Cuzinatto, C.A.M. de Melo, P.J. Pompeia, Ann. Phys. **322**, 1211 (2007)
22. A.I. Alekseev, B.A. Arbuzov, V.A. Baikov, Theor. Math. Phys. **52**, 739 (1982)
23. S.A. Fulling, R.C. King, B.G. Wybourne, C.J. Cummings, Class. Quantum Grav. **9**, 1151 (1992)
24. Y. Décanini, A. Folacci, Class. Quantum Grav. **24**, 4777 (2007)
25. M. Blagojevic, Gravitation & Gauge Symmetries (Taylor & Francis, London, 2001)
26. V. De Sabbata, M. Gasperini, Introduction to Gravitation (World Scientific, Singapore, 1985)
27. I.G. Avramidi, Heat Kernel and Quantum Gravity (Springer, Berlin, 2000)